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Coupled integral equation approach for solving melting or solidification

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Abstract—A coupled integral equation formulation is developed to obtain an analytic expression for the location of the moving interface in planar melting or solidification of a semi-infinite medium at the melting point, subjected to a time-varying surface temperature. The low-order solution of the present analysis is compared with the exact results for the case of constant surface temperature, and with the finite-difference predictions for the case of time-varying surface temperature. The results are sufficiently accurate over the range of Stefan number up to about 5, which covers almost all materials of practical interest.

INTRODUCTION

THE SOLUTION of moving boundary problems involving melting or solidification is inherently difficult because the location of the moving interface is not known *a priori* and must follow as a part of the solution. Because of nonlinearity of the governing equations, exact analytic solutions can be developed only for a limited number of specific cases which are well documented in the literature [1, 2]. Therefore, considerable amount of effort has been directed towards the development of approximate analytic as well as numerical solution techniques. The available approximate analytic solution techniques include, among others, the integral method [3, 4], the variational formulation [5, 6], the moving heat source approach [7–10], the perturbation technique [11–13], the embedding technique [14–16] and the variable eigenvalue approach [17]. A variety of purely numerical methods of solution have also been reported [18–22].

Most of these approximate analytic techniques have difficulty in accommodating time-dependent applied surface conditions, rely on the choice of a suitable approximate profile for the temperature distribution, and their applicability is restricted to short times.

In the analysis of phase-change problems, a quantity of practical interest is the location of the moving interface as a function of time. In this work, we present a coupled integral equation formulation of the phase-change problem in order to develop an analytic expression for the location of the moving interface resulting from an applied surface temperature which varies with time.

TRANSFORMATION TO INTEGRAL EQUATIONS

Consider a semi-infinite medium initially at its melting point subjected to melting or solidification resulting from an applied surface temperature which varies with time. The mathematical formulation of this phase-change problem is given in the dimensionless form as

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad \text{in} \quad 0 < x < s(t), \quad t > 0 \tag{1a}$$

$$u(x,t) = F(t) \quad \text{at} \quad x = 0, \quad t > 0 \tag{1b}$$

$$u(x,t) = 1 \quad \text{at} \quad x = s(t), \quad t > 0 \tag{1c}$$

which is coupled to the following interface energy-balance equation

$$\frac{\partial u(x,t)}{\partial x} = \frac{1}{\Lambda}\frac{\mathrm{d}s(t)}{\mathrm{d}t} \quad \text{at} \quad x = s(t), \quad t > 0 \tag{2a}$$

$$s(t) = 0 \quad \text{for} \quad t = 0. \tag{2b}$$

To transform equation (1a) into integral form, we introduce a new variable $v(x,t)$, defined as

$$v(x,t) = \frac{\partial u(x,t)}{\partial x} \tag{3a}$$

hence we can write

$$\frac{\partial v(x,t)}{\partial x} = \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t}. \tag{3b}$$

Then, the energy equation (1a) is expressed in the form

$$\frac{\partial u(x,t)}{\partial x} = v(x,t) \tag{4a}$$

$$\frac{\partial v(x,t)}{\partial x} = \dot{u}(x,t). \tag{4b}$$

We integrate equations (4a) and (4b) from $x = 0$ to $s(t)$ and obtain, respectively

$$u(s,t) - u(0,t) = \int_0^{s(t)} v(x,t)\,\mathrm{d}x \tag{5a}$$

NOMENCLATURE

b	reference length
C_p	specific heat
$f(t^*)$	applied surface temperature
f_0	value of applied surface temperature at time $t^* = 0, f(0)$
L	latent heat of solidification
$s^*(t^*)$	location of moving interface
$s(t)$	dimensionless location of moving interface, $s^*(t^*)/b$
t^*	time
t	dimensionless time, $\alpha t^*/b^2$
$T(x^*, t^*)$	temperature
T_m	melt temperature
$u(x,t)$	dimensionless temperature, $\frac{T(x^*,t^*)-f_0}{T_m-f_0}$
x^*	physical coordinate
x	dimensionless coordinate, x^*/b.

Greek symbols

α	thermal diffusivity
Λ	Stefan number, $\frac{C_p(T_m-f_0)}{L}$.

$$v(s,t)-v(0,t) = \int_0^{s(t)} \dot{u}(x,t)\,dx = \frac{d}{dt}\left[\int_0^{s(t)} u(x,t)\,dx\right] - u(s,t)\dot{s}(t). \quad (5b)$$

Equations (1b, c) and (2a, b) are rewritten as

$$u(0,t) = F(t) \quad (6a)$$

$$u(s,t) = 1 \quad (6b)$$

$$\frac{\partial u(s,t)}{\partial x} = \frac{1}{\Lambda}\dot{s}(t) \quad (7a)$$

$$s(0) = 0. \quad (7b)$$

where the dot over u or s refers to derivative with respect to the time variable.

Equations (4), (6) and (7) are utilized to recast equations (5a, b) in the form

$$1-F(t) = \int_0^{s(t)} v(x,t)\,dx \quad (8a)$$

$$\frac{\dot{s}(t)}{\Lambda} - v(0,t) = \frac{d}{dt}\left[\int_0^{s(t)} u(x,t)\,dx\right] - \dot{s}(t). \quad (8b)$$

These two integral expressions, together with the interface energy balance equation (7), provide three relations for the determination of the three unknowns $u(x,t)$, $v(x,t)$ and $s(t)$. However, our interest is in the determination of $s(t)$ only.

So far our analysis is exact. In addition, equations (8) are in such a convenient form that the integrals can readily be approximated directly by suitable representations involving $v(x,t)$, $u(x,t)$ and their derivatives evaluated at the two end points of the region $0 \leqslant x \leqslant s(t)$. The end point values can then be related to $s(t)$ by making use of the boundary conditions and the differential equations.

A LOW-ORDER ANALYSIS

To approximate the integrals given by equations (8), we choose the trapezoidal rule over the region $0 \leqslant x \leqslant s(t)$ given by

$$\int_0^s y(x)\,dx \cong \frac{s}{2}[y(0)+y(s)]. \quad (9)$$

Although this expression implies a linear variation of $v(x,t)$ when applied to equation (8a), the variation of $u(x,t)$ with x is quadratic because of the definition of $v(x,t)$ given by equation (4a). However, the use of trapezoidal rule in equation (8b) implies a linear variation for $u(x,t)$ in that equation.

The representation (9) is now applied to equation (8) to remove the integrals, and the relations (4), (6) and (7) are utilized to eliminate the resulting terms $v(s,t)$ and $u(s,t)$. Then equations (8) take the form

$$1-F(t) = \frac{1}{2\Lambda}s(t)\dot{s}(t) + \tfrac{1}{2}s(t)v(0,t) \quad (10a)$$

$$\frac{\dot{s}(t)}{\Lambda} - v(0,t) = \tfrac{1}{2}[1+F(t)]\dot{s}(t) + \dot{F}(t)\frac{s(t)}{2} - \dot{s}(t). \quad (10b)$$

The function $v(0,t)$ is eliminated between equations (10a, b).

$$\frac{1}{\Lambda}\left[\left(1+\frac{\Lambda}{4}\right) - \frac{\Lambda}{4}F(t)\right]s(t)\dot{s}(t) - \frac{\dot{F}(t)}{4}s^2(t) = 1-F(t). \quad (11)$$

Now, a new variable $\sigma(t)$ is defined as

$$\sigma(t) = [s(t)]^2. \quad (12)$$

Then equation (11) takes the form

$$\frac{1}{2\Lambda}\left[\left(1+\frac{\Lambda}{4}\right) - \frac{\Lambda}{4}F(t)\right]\dot{\sigma}(t) - \tfrac{1}{4}\dot{F}(t)\sigma(t) = 1-F(t) \quad (13a)$$

and the initial condition (7b) becomes

$$\sigma(0) = 0. \quad (13b)$$

Thus the phase-change problem is now reduced to the solution of an ordinary differential equation for the function $\sigma(t)$. Here $F(t)$ is a known boundary condition function. Once $\sigma(t)$ is known, the location of the solid–

liquid interface is determined according to equation (12).

The differential equation (13) is solved and the following expression is obtained for the location of the moving interface:

$$s(t) = \frac{\left\{2\Lambda \int_0^t [1-F(\tau)]\left[1+\frac{\Lambda}{4}(1-F(\tau)\right]d\tau\right\}^{1/2}}{1+\frac{\Lambda}{4}[1-F(t)]}. \quad (14)$$

In the case of constant applied surface temperature we have $f(t^*) = f_0 = \text{constant}$. Then, from the definition of the dimensionless applied surface temperature we obtain $F(t) = 0$. Thus, for the case of constant applied surface temperature, the solution (14) reduces to

$$s(t) = 2\eta_{app}\sqrt{t} \quad (15a)$$

where

$$\eta_{app} = \sqrt{\frac{2\Lambda}{4+\Lambda}}. \quad (15b)$$

The accuracy of these solutions is discussed next.

ACCURACY OF THE SOLUTIONS

The phase-change problem considered above has an exact solution for the case of constant applied surface temperature. Therefore, the solution (15) can be compared directly with the exact results.

In the case of time-dependent applied surface temperature, no exact analytic solution is available. Therefore, for such cases the problem is solved numerically by using a finite-difference scheme and the numerical results are used for checking the accuracy of approximate solutions.

Constant surface temperature

When the applied surface temperature is constant, the problem admits exact solution and the resulting expression for the location of the moving interface is given by [2, p. 410]

$$s(t) = 2\eta\sqrt{t} \quad (16)$$

where the exact value of η is determined from the solution of the following transcendental equation

$$\sqrt{\pi}\eta\, e^{\eta^2}\, \text{erf}(\eta) = \Lambda. \quad (17)$$

In Table 1 we present the percentage error involved in the approximate solution given by equations (15). For the values of the Stefan number up to about 5, which covers almost all the materials of practical interest, the solution yields results within 2% or less of the exact value. This is sufficiently accurate for most practical applications. Furthermore, the solution provides a very simple expression for the location of the moving interface.

Table 1. The accuracy of approximate solution [equation (15)] for the case of constant surface temperature

Stephan number, Λ	η (exact)	% Error $[(\eta_{app}-\eta)/\eta] \times 100$
0.02	0.099668	0.001
0.1	0.22001	0.386
0.2	0.30642	0.715
1	0.62006	2.00
2	0.80060	1.99
5	1.05968	0.528
10	1.25697	−4.91

Time-dependent surface temperature

When the applied surface temperature varies with time, the functional form of $F(t)$ effects the motion of the solid–liquid interface. Suppose $F(t)$ varies linearly in the form

$$F(t) \equiv \frac{f(t^*)-f_0}{T_m-f_0} = a+bt. \quad (18)$$

Then, introducing equation (18) into (14) we obtain the following explicit expression for the location $s(t)$ of the moving interface:

$$s(t) = (8\Lambda)^{1/2}\left\{\left[(1-a)[4+\Lambda(1-a)]t - b[2+\Lambda(1-a)]t^2 + \frac{\Lambda b^2}{3}t^3\right]^{1/2}\right\} \div \{4+\Lambda(1-a-bt)\}. \quad (19)$$

To check the accuracy of this approximate analytic solution, we considered the following specific case

$$F(t) = 0.2t \quad (20a)$$

which implies that the surface temperature is suddenly lowered below the melt temperature T_m and then gradually raised as a linear function of time. The freezing continues as long as the surface temperature remains below the melt temperature T_m. For this particular case, by setting $a = 0$ and $b = 0.2$, equation (19) reduces to

$$s(t) = (8\Lambda)^{1/2} \times \frac{\left\{(4+\Lambda)t - 0.2(2+\Lambda)t^2 + \frac{0.04}{3}\Lambda t^3\right\}^{1/2}}{4+\Lambda(1-0.2t)}. \quad (20b)$$

To examine the accuracy of equation (20b), we solved this phase-change problem numerically by using a finite-difference scheme involving a variable time step method as discussed in ref. [22]. The numerical calculations were performed with a step size $\Delta x = 0.005$, while a much finer step size $\Delta x_0 = 10^{-5}$ was used for starting. The calculations were stopped when the applied surface temperature equals to T_m.

Table 2 shows a comparison of $s(t)$ determined from the approximate analytic solution given by equation (20b) and from the finite-difference calculations, for

Table 2. A comparison of $s(t)$ determined from equation (20b) and the finite-difference solution for an applied surface temperature varying as $F(t) = 0.2t$

Stefan number, Λ	time, t	$s(t)$ from finite-difference	$s(t)$ from equation (20b)
0.2	0.6239 − 02	0.5000 − 01	0.4874 − 01
	0.2618 − 01	0.1000	0.9975 − 01
	0.1068	0.2000	0.2008
	0.2441	0.3000	0.3017
	0.4425	0.4000	0.4027
	0.7094	0.5000	0.5037
	0.1057 + 01	0.6000	0.6048
	0.1508 + 01	0.7000	0.7062
	0.2103 + 01	0.8000	0.8079
	0.2951 + 01	0.9000	0.9104
1.0	0.6206 − 02	0.1000	0.9964 − 01
	0.2561 − 01	0.2000	0.2023
	0.5816 − 01	0.3000	0.3047
	0.1040	0.4000	0.4071
	0.1634	0.5000	0.5096
	0.2367	0.6000	0.6124
	0.3242	0.7000	0.7154
	0.4265	0.8000	0.8188
	0.5443	0.9000	0.9227
	0.6783	0.1000 + 01	0.1027 + 01
5.0	0.1964 − 02	0.1000	0.9343 − 01
	0.8496 − 02	0.2000	0.1944
	0.1949 − 01	0.3000	0.2945
	0.3495 − 01	0.4000	0.3946
	0.5490 − 01	0.5000	0.4949
	0.7935 − 01	0.6000	0.5954
	0.1083	0.7000	0.6964
	0.1418	0.8000	0.7978
	0.1799	0.9000	0.8997
	0.2226	0.1000 + 01	0.1002 + 01

three different values of the Stefan number, $\Lambda = 0.2$, 1 and 5. For the cases considered in this table, results from equation (20b) are within 3% of the numerical predictions, which is sufficiently accurate for most practical purposes. We repeated the calculations by using $F(t) = -0.2t$; the accuracy remained essentially the same.

CONCLUSIONS

In the present method of analysis, the differential equation of heat conduction is directly transformed into two coupled integral equations. Such equations are most convenient to approximate the integrals with suitable representations over the domain $0 \leqslant x \leqslant s(t)$, thus alleviating the need for finding suitable profiles for the temperature distribution in the medium. The accuracy and the usefulness of the method is tested by using a trapezoidal rule representation of the integrals. Resulting analytic expressions for the location of the moving interface are very simple, yet sufficiently accurate for most practical applications. In this work, the method is applied to the solution of solidification (or melting) in a semi-infinite planar medium initially at the phase-change temperature, subjected to time-dependent applied surface temperature; but it is sufficiently general for extension to the solution of more involved phase-change problems.

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UNE APPROCHE D'EQUATION INTEGRALE COUPLEE POUR TRAITER LA FUSION OU LA SOLIDIFICATION

Résumé—Une formulation d'équation intégrale couplée est développée pour obtenir une expression analytique pour la localisation de l'interface mobile dans la fusion ou la solidification plane d'un milieu semi-infini au point de fusion, soumis à une température de surface variable. La solution, d'ordre faible, de cette analyse est comparée avec les résultats exacts dans le cas d'une température de surface constante, et avec les prévisions par différences finies dans le cas de la température de surface suffisamment précis dans le domaine de nombre de Stephan jusqu'à 5 environ, ce qui couvre la plupart des matériaux d'intérêt pratique.

DIE LÖSUNG VON SCHMELZ- UND VERFESTIGUNGSPROBLEMEN MIT HILFE GEKOPPELTER INTEGRALGLEICHUNGEN

Zusammenfassung—Es wurde ein System gekoppelter Integralgleichungen entwickelt, um einen analytischen Ausdruck für den Ort der bewegten Phasengrenze beim ebenen Schmelzen oder Verfestigen eines halbunendlichen Mediums von Schmelztemperatur zu erhalten, wenn an der Oberfläche eine zeitveränderliche Temperatur aufgeprägt wird. Die Lösung niedriger Ordnung wird mit den genauen Ergebnissen für den Fall konstanter Oberflächentemperatur und mit den Berechnungen nach dem Finite-Differenzen-Verfahren für den Fall von zeitveränderlicher Oberflächentemperatur verglichen. Die Ergebnisse sind für Stefan-Zahlen bis zu ungefähr 5 ausreichend genau, was fast alle Materialien von praktischem Interesse abdeckt.

ПРИМЕНЕНИЯ ИНТЕГРАЛЬНОГО УРАВНЕНИЯ ДЛЯ РЕШЕНИЯ ЗАДАЧ ПЛАВЛЕНИЯ ИЛИ ЗАТВЕРДЕВАНИЯ

Аннотация—Выведено связаннос интегральное уравнение с целью получения аналитического выражения для нахождения движущейся границы при плоском плавлении или затвердевании полуограниченной среды при изменяющейся по времени температуре поверхности. Полученное решение сравнивается с точными результатами для случая постоянной температуры поверхности и с расчетами по методу конечных резностей для случая изменяющейся по времени температуры поверхности. Имеет место достаточно хорошее совпадение результатов в диапазоне изменения числа Стефана вплоть до 5 почти для всех материалов, имеющих практический интерес.